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Vertex-transitive expansions of (1, 3)-trees

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ABSTRACT

A nonidentity automorphism of a graph is said to be *semiregular* if all of its orbits are of the same length. Given a graph X with a semiregular automorphism γ, the quotient of X relative to γ is the multigraph X/γ whose vertices are the orbits of γ and two vertices are adjacent by an edge with multiplicity r if every vertex of one orbit is adjacent to r vertices of the other orbit. We say that X is an *expansion* of X/γ. In [J.D. Horton, I.Z. Bouwer, Symmetric Y-graphs and H-graphs, J. Combin. Theory Ser. B 53 (1991) 114–129], Horton and Bouwer considered a restricted sort of expansions (which we will call 'strong' in this paper) where every leaf of X/γ expands to a single cycle in X. They determined all cubic arc-transitive strong expansions of simple (1, 3)-trees, that is, trees with all of their vertices having valency 1 or 3, thus extending the classical result of Frucht, Graver and Watkins (see [R. Frucht, J.E. Graver, M.E. Watkins, The groups of the generalized Petersen graphs, Proc. Cambridge Philos. Soc. 70 (1971) 211–218]) about arc-transitive strong expansions of K_2 (also known as the generalized Petersen graphs). In this paper another step is taken further by considering the possible structure of cubic vertex-transitive expansions of general (1,3)-multitrees (where vertices with double edges are also allowed); thus the restriction on every leaf to be expanded to a single cycle is dropped.

1. Introductory remarks

Throughout this paper, the graphs are finite, connected, undirected and simple, unless specified otherwise. For adjacent vertices u and v in X, we write $u \sim v$ and denote the corresponding edge by uv. Given a graph X we let $V(X)$ and $\mathrm{Aut}(X)$ be the vertex set and the automorphism group of X, respectively. A graph X is said to be *vertex-transitive* if its automorphism group $\mathrm{Aut}(X)$ acts transitively on $V(X)$. If uv is an edge of the graph X, then (u, v) and (v, u) are the two arcs of X associated with uv. A graph X is *arc-transitive* (or, equivalently, *symmetric*) if $\mathrm{Aut}(X)$ acts transitively on the arcs of X.

A permutation of a finite set is called (m, n)-*semiregular*, where $m \geq 1$ and $n \geq 2$ are integers, if it has m orbits of length n and no other orbit. Let X be a graph with an (m, n)-semiregular automorphism γ. One may then define a natural quotient multigraph X/γ whose vertex set consists of the orbits of γ, with two orbits being joined by an edge with multiplicity r if every vertex of one orbit is adjacent to r vertices of the other orbit. In particular, when X is a cubic graph then the vertices of X/γ may have valencies 1, 2 or 3 where vertices of valency 2 may only occur if n is even, with each orbit corresponding to one such vertex inducing $\frac{n}{2}K_2$. Alternatively, one may view a cubic graph X with an (m, n)-semiregular automorphism γ as a cubic *expansion* of its quotient graph X/γ, where each leaf (a vertex of valency 1) of X/γ is replaced by an n-circulant of valency 2, a vertex of valency 2 of X/γ is replaced by $\frac{n}{2}K_2$, and a vertex of valency 3 of X/γ is replaced by n independent

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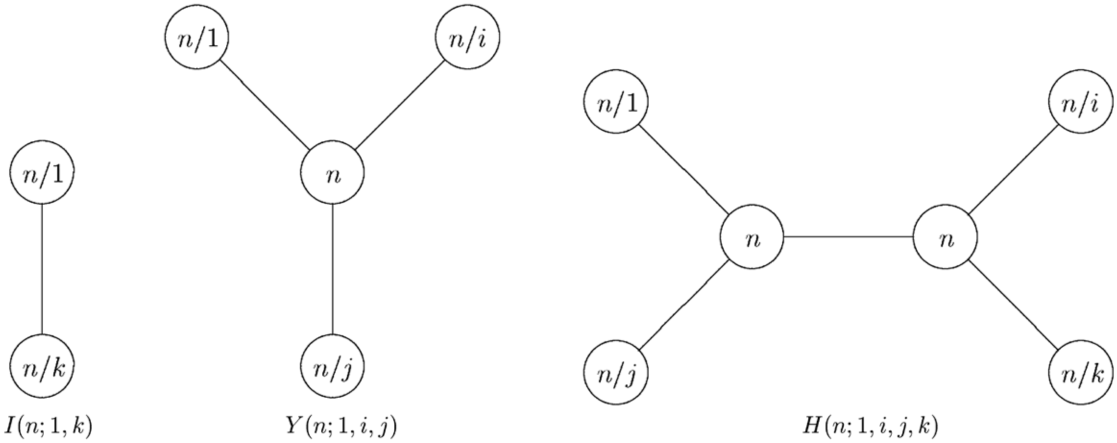

Fig. 1. Generalized Petersen graphs, Y-graphs and H-graphs in Frucht's notation.

vertices. Of course, these replacements must all be done in such a way that the resulting graph admits the existence of a corresponding semiregular automorphism. However, the quotients X/γ with vertices of valency 2 will not be dealt with in the rest of this paper. Yet another popular notion can be applied here, namely, viewing X/γ as a *base graph* and X as a *regular covering graph* over X/γ with the addition of loops to the leaves of X/γ and with suitable voltages on the edges (see, for example, [10, Chapter 2]).

A long-standing conjecture stating that every vertex-transitive graph has a semiregular automorphism [14] has recently received a renewed interest (see [3–5,7–9,15,17]). In particular, it is known to hold for cubic and quartic graphs. Hence the structure of cubic vertex-transitive graphs may be studied via their semiregular automorphisms and the corresponding natural quotients. For example, consider the case where the quotient in question is the dipole Dip_3 consisting of two vertices with three parallel edges. It is not difficult to see that the starting (covering) graph is then a Cayley graph of a dihedral group. Thus, the problem of determining arc-transitive expansions of Dip_2 translates to the problem of classifying cubic arc-transitive Cayley graphs of dihedral groups. This was done, in a somewhat different context, in [16]. Much earlier, the classification of all arc-transitive generalized Petersen graphs $GP(n, k) = I(n; 1, k)$, that is, arc-transitive strong expansions of K_2 (see Fig. 1), was obtained by Frucht, Graver and Watkins [6]. The notation $I(n; k, l)$ is taken from Boben, Pisanski and Žitnik [2] where the broader class of *I-graphs* that includes the generalized Petersen graphs was introduced. Further, in [11], Horton and Bouwer generalized this result by determining all arc-transitive cubic graphs arising as strong expansions from simple $(1, 3)$-trees, that is, trees with vertex valencies 1 or 3 only. They proved that only two additional simple trees, other than K_2, can be such quotients: the Y-tree $K_{1,3}$ and the H-tree consisting of two vertices of valency 3 and four vertices of valency 1 (leaves). More precisely, letting $Y(n; 1, i, j)$ and $H(n; 1, i, j, k)$ denote the last two graphs given in Fig. 1 via Frucht's notation, we may formulate their result as follows. (We refer to these graphs also as *Y-graphs* and *H-graphs*.)

Proposition 1.1 (*Horton and Bouwer [11]*). *Let X be a cubic arc-transitive graph and γ a semiregular automorphism of X relative to which the quotient graph X/γ is a simple $(1, 3)$-tree, such that X is a strong expansion of X/γ. Then one of the following occurs:*

(i) *X is an expansion of K_2 and isomorphic to one of the seven arc-transitive generalized Petersen graphs: $I(4; 1, 1)$, $I(5; 1, 2)$, $I(8; 1, 3)$, $I(10; 1, 2)$, $I(10; 1, 3)$, $I(12; 1, 5)$ or $I(24; 1, 5)$ [6];*
(ii) *X is a strong expansion of the Y-tree $K_{1,3}$ and isomorphic to one of the following four arc-transitive graphs: $Y(7; 1, 2, 4)$, $Y(14; 1, 3, 9)$, $Y(28; 1, 3, 9)$ or $Y(56; 1, 9, 25)$;*
(iii) *X is a strong expansion of the H-tree and isomorphic either to $H(17; 1, 4, 2, 8)$ or to $H(34; 1, 13, 9, 15)$.*

We remark that $Y(7; 1, 2, 4)$ is the well-known Coxeter graph and that $H(17; 1, 4, 2, 8)$ is the smallest member of an infinite family of primitive cubic graphs associated with the action of the group $PSL(2, p)$, where $p \equiv 1 (\mathrm{mod}\ 16)$, on cosets of Sym_4, with $p = 17$.

Staying within the $(1, 3)$-multitrees realm for possible quotients, this leads us to the natural generalization of the problem considered by Horton and Bouwer where double edges in the quotients of the graphs in question are allowed, as well as expansions in general, not only the strong ones. To this end, following the examples of I-graphs, Y-graphs and H-graphs, we define $(1, 3)$-*graphs* as follows:

Definition 1.2. A graph X is called a $(1, 3)$-*graph* if it is cubic, admits a semiregular automorphism γ, and its quotient X/γ is a $(1, 3)$-tree.

Notice that by [17], every cubic vertex-transitive graph admits a semiregular automorphism. The aim of this paper is therefore to consider vertex-transitive, and in particular arc-transitive, $(1, 3)$-graphs which are expansions of arbitrary $(1, 3)$-multitrees having not only single but also double edges.

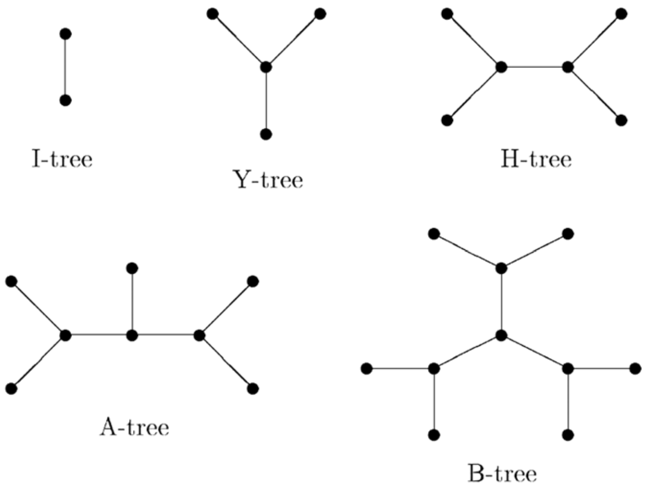

Fig. 2. Simple (1, 3)-trees as quotients.

Let us briefly touch the case of non-strong expansions of simple (1, 3)-trees. According to [11], only I-tree, Y-tree, H-tree, A-tree, or B-tree, can be a quotient of a cubic arc-transitive graph over a semiregular automorphism (see Fig. 2). In [13, Section 3], the symmetry of I-graphs has been considered more in detail from which it follows that there are no arc-transitive I-graphs other than those seven from Proposition 1.1(i). A slight modification of the argument at the end of [11, p. 128] shows that there are no arc-transitive non-strong expansions of A-tree and B-tree either. However, the question of arc-transitive non-strong expansions of Y- and H-tree remains open.

If the quotient is indeed a (1, 3)-multitree with at least one double edge, then we can prove a somewhat stronger result involving vertex-transitive (1, 3)-graphs in general (not only arc-transitive ones). The following theorem is proved in this paper over a series of propositions:

Theorem 1.3. *Let X be a vertex-transitive (1, 3)-graph and γ a semiregular automorphism of X relative to which the quotient graph X/γ is a (1, 3)-multitree with at least one double edge. Then X/γ is a path of odd length with alternating single and double edges.*

The theorem above, together with those of Frucht, Graver and Watkins [6], and Horton and Bouwer [11], give us the main result of this paper.

Theorem 1.4. *Let X be an arc-transitive (1, 3)-graph and γ a semiregular automorphism of X relative to which the quotient graph X/γ is a (1, 3)-multitree. Moreover, suppose that X is a strong expansion whenever X/γ is a Y-tree or a H-tree. Then one of the following occurs:*

(i) *X is a expansion of K_2 (an I-graph) and isomorphic to one of the seven arc-transitive generalized Petersen graphs: $I(4;\ 1,\ 1)$, $I(5;\ 1,\ 2)$, $I(8;\ 1,\ 3)$, $I(10;\ 1,\ 2)$, $I(10;\ 1,\ 3)$, $I(12;\ 1,\ 5)$ or $I(24;\ 1,\ 5)$;*
(ii) *X is a strong expansion of the Y-tree $K_{1,3}$ (a Y-graph) and isomorphic to one of the following four arc-transitive graphs: $Y(7;\ 1,\ 2,\ 4)$, $Y(14;\ 1,\ 3,\ 9)$, $Y(28;\ 1,\ 3,\ 9)$ or $Y(56;\ 1,\ 9,\ 25)$;*
(iii) *X is a strong expansion of the H-tree (a H-graph) and isomorphic either to $H(17;\ 1,\ 4,\ 2,\ 8)$ or to $H(34;\ 1,\ 13,\ 9,\ 15)$;*
(iv) *X/γ is a path of odd length ≥ 3, whose every internal vertex is incident to one single and one double edge.*

In Section 2 we give examples of graphs satisfying part (iv) of Theorem 1.4. The proof of Theorems 1.3 and 1.4 is then carried out in Sections 3 and 4. The following notational conventions are in order in that respect.

Throughout the rest of this paper, X stands for a (finite) cubic vertex-transitive graph with $G = \mathrm{Aut}(X)$ denoting its group of automorphisms, and $\gamma \in G$ will be a (fixed) semiregular automorphism whose vertex-orbits have cardinality $n \geq 3$. As remarked, such automorphism exists by [17]. The corresponding quotient graph, a (1, 3)-multitree, will be denoted by the symbol $\mathcal{X} = X/\gamma$. Further, in view of the results of Horton and Bouwer (see Proposition 1.1 and the subsequent discussion), we may assume that $\mathcal{X}$ necessarily contains double edges. Let us denote by $\mathcal{X}^*$ the graph obtained from $\mathcal{X}$ by removing all double edges. Since X is cubic, it follows that $\mathcal{X}^*$ is a collection of simple (1, 3)-trees. This situation is covered in the last two sections. Section 3 deals with graphs whose quotients are not paths, and excludes the possibility of occurrence of various components of $\mathcal{X}^*$ isomorphic to one of the following: the A-tree, the B-tree, or the H-tree. Finally, in Section 4, the Y-trees as components of $\mathcal{X}^*$ are excluded, thus completing the proof of Theorems 1.3 and 1.4.

Instead of $a \equiv b \pmod{n}$, the more concise notation $a \equiv_n b$ is used throughout this paper.

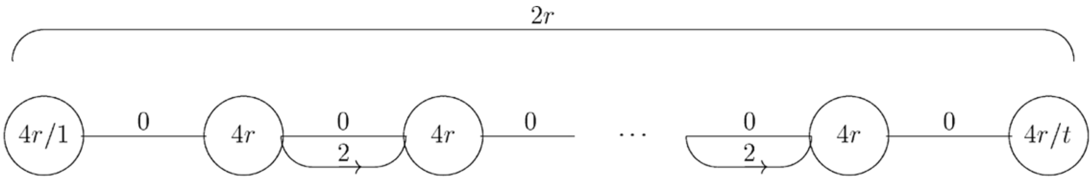

Fig. 3. $I_{2r}^{4r}(t)$-path, $t = 2r + 1$.

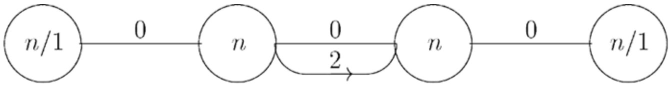

Fig. 4. $I_4^n(1)$-path.

2. Examples

In this section, we give two examples which fall into Case (iv) of Theorem 1.4. The first one is a family of symmetric (arc-transitive) graphs which are described in Theorem 1.1 of [12]. Such a graph is called an $I_{2r}^{4r}(t)$-*path* where $r = 2, 3, 4, \ldots$ and $t = 2r + 1$ (see also [12, section 2.4]), and should not be confused with I-graphs defined in [2]. It has a semiregular automorphism γ of order $4r$ and is depicted in Fig. 3.

The second example is a family of $I_4^n(1)$-paths where $n = 4m + 2$ and $m = 1, 2, 3, \ldots$. Denote the $I_4^n(1)$-path (Fig. 4) shortly by X_n; thus, X_n admits a semiregular automorphism γ of order n. The two leaves of X_n/γ expand to two n-cycles in X_n; also, the double edge in the middle expands to two additional n-cycles. There exists an automorphism α of X_n which acts cyclically on these four n-cycles, so α is of order 4. It is not difficult to verify that X_n is a Cayley graph for the group generated by γ and α with a relation $\gamma^{-1}\alpha\gamma = \alpha^{-1}$ where the generators are chosen as $\{\gamma, \alpha\gamma^{n/2}\}$; hence, X_n is vertex-transitive. Let us denote the left leaf of X_n/γ by y, its neighbour by x, the other neighbour of x by z, and the right leaf by t. Also, in X_n, let the expanded vertices be denoted by $y_0, \ldots, y_{n-1}, x_0, \ldots, x_{n-1}, z_0, \ldots, z_{n-1}$ and $t_0, \ldots, t_{n-1}$, respectively. Now, if $n = 6$, then the edge y_0x_0 is on two 6-cycles:

$$y_0x_0z_0x_4y_4y_5 \quad \text{and} \quad y_0x_0z_2x_2y_2y_1$$

but y_0y_1 is on three 6-cycles

$$y_0y_1y_2x_2z_2x_0, \quad y_0y_1x_1z_1x_5y_5 \quad \text{and} \quad y_0y_1y_2y_3y_4y_5.$$

For $n \geq 10$, note that y_0x_0 is on at least four 8-cycles:

$$y_0x_0z_0t_0t_1z_1x_1y_1, \qquad y_0x_0z_0t_0t_{n-1}z_{n-1}x_{n-1}y_{n-1},$$
$$y_0x_0z_2t_2t_3z_3x_1y_1, \qquad y_0x_0z_2t_2t_1z_1x_{n-1}y_{n-1},$$

whereas x_0z_0 is on just two such cycles:

$$x_0z_0t_0t_1z_1x_1y_1y_0 \quad \text{and} \quad x_0z_0t_0t_{n-1}z_{n-1}x_{n-1}y_{n-1}y_0.$$

Thus, X_n is not edge-transitive.

3. Excluding various $\mathcal{X}^*$-components

In this and the next section we suppose that $\mathcal{X}^*$ consists of at least two components, and let C^* be an arbitrary component of $\mathcal{X}^*$. Then X contains a subgraph C which is isomorphic to C^* and is obtained as follows: choose a vertex x in a γ-orbit $\mathcal{O}$ such that $\mathcal{O}$ is a vertex of C^*, and traverse all paths starting in x and containing edges which belong to edge-orbits in C^*.

Proposition 3.1. *If C^* is a component of $\mathcal{X}^*$ and C is obtained from C^* using the procedure described above, then there exists an involution $\alpha \in G$ fixing every vertex of C.*

Proof. Let $\mathcal{O}_x = \{x_0, \ldots, x_{n-1}\}$ be a γ-orbit which is a vertex of C^* and define $\alpha(x_i) = x_{-i}$. Let $\mathcal{O}_y = \{y_0, \ldots, y_{n-1}\}$ be another vertex in C^* such that x_0 and y_0 are adjacent, and $x_0, y_0 \in V(C)$. If we extend the definition of α by $\alpha(y_i) = y_{-i}$, it is clear that α is one-to-one and preserves adjacency in $\mathcal{O}_x \cup \mathcal{O}_y$. Hence α can be extended to the whole of C^*; in particular, α is an involution that fixes every vertex of C.

To extend α to the whole of X we proceed as follows. Let $\mathcal{O}_z = \{z_0, \ldots, z_{n-1}\}$ be a γ-vertex-orbit of X such that there is a double edge-orbit between $\mathcal{O}_x$ and $\mathcal{O}_z$ in $\mathcal{X}$. More precisely, x_0 is adjacent to z_0 and also to z_k for some integer k. Now define $\alpha(z_i) = z_{k-i}$. Obviously, α is still an involution which preserves adjacencies between $\mathcal{O}_x$ and $\mathcal{O}_z$. Therefore, it is easy to extend α to every component of $\mathcal{X}^*$, that is, to the whole of X. Because $\mathcal{X}$ is a tree, α is a well-defined automorphism of X. ■

Proposition 3.2. *If $\mathcal{X}$ is not a path, then X is edge-transitive.*

Proof. Suppose X is not edge-transitive, let $x \in V(X)$ and $\mathcal{O}$ be the γ-orbit of X containing x. Furthermore, choose x and $\mathcal{O}$ such that $\mathcal{O}$ (as a vertex in $\mathcal{X}$) is incident to a double edge (in $\mathcal{X}$). Now, X is vertex-transitive but not edge-transitive, hence the three edges (say, e, e', e'') incident to x belong to more than one G-orbit. However, according to Proposition 3.1, there is an $\alpha \in G$ fixing x and swapping two of the incident edges (for instance, e' and e''). Also, if $\mathcal{E}', \mathcal{E}''$ are the two γ-orbits containing e' and e'', respectively, then $\mathcal{E}'$ and $\mathcal{E}''$ form a double edge in $\mathcal{X}$. Thus e' and e'' are in the same G-orbit which we will denote by $\mathcal{E}_2$, and e belongs to the other orbit (for instance, $\mathcal{E}_1$). Hence the whole set $E(X)$ of edges of X breaks into two G-orbits. The above reasoning also shows that for any double edge of $\mathcal{X}$, the corresponding edges of X are all members of $\mathcal{E}_2$ and not of $\mathcal{E}_1$. It remains to figure out what happens to those edges of X contained in γ-orbits which represent single edges in $\mathcal{X}$. Let $\mathcal{E} = \{e_0, \ldots, e_{n-1}\}$ be such a γ-orbit. If $\mathcal{E}$ is adjacent to a double edge in $\mathcal{X}$, then clearly $\mathcal{E} \subseteq \mathcal{E}_1$. Likewise, if $\mathcal{E}$ is incident to a *leaf* in $\mathcal{X}$, then $\mathcal{E} \subseteq \mathcal{E}_1$, too. Therefore, let $\mathcal{E}$ be neither incident to a leaf nor adjacent to a double edge. Assume that $\mathcal{E} \subseteq \mathcal{E}_2$; then there is another single edge of $\mathcal{X}$ adjacent to $\mathcal{E}$ that belongs to $\mathcal{E}_2$. Continuing this way we obtain a path of single edges of $\mathcal{X}$ contained entirely in $\mathcal{E}_2$. But this path is finite because $\mathcal{X}$ is finite, and it ends with a single edge that is either adjacent to a double edge, or has a leaf as an endvertex. As this is contradictory, it follows that $\mathcal{E} \subseteq \mathcal{E}_1$, but this implies that all edges adjacent to $\mathcal{E}$ in $\mathcal{X}$ are in the other orbit $\mathcal{E}_2$ which is only possible if $\mathcal{X}$ itself is a path of alternating single and double edges. ■

We now turn our attention to graphs whose quotients are not paths. If $\mathcal{X}$ is not a path then, by Proposition 3.2, X is edge-transitive. But X is also a cubic vertex-transitive graph, hence it must be symmetric (1-arc-transitive) by the well-known theorem of W.T. Tutte. Another theorem of Tutte tells us that X is at most q-arc-transitive where q is not greater than 5. However, it follows from Proposition 3.1 that for every component C^* of $\mathcal{X}^*$ there is an involution $\alpha \in G$ such that α fixes the underlying subgraph C of X vertex-wise. Let P be a maximal path in C. Then it follows from [1, Proposition 18.1] that the length of P must be less than $q \leq 5$, thus the diameter of C is at most 4 which leaves us with just five possibilities for components C^*, which will be named *I-tree*, *Y-tree*, *H-tree*, *A-tree*, and *B-tree*, respectively (see Fig. 2).

Note that the I-tree is actually K_2 and the Y-tree is $K_{1,3}$. We will dispose of all except one of the aforementioned possibilities (cf. [11]). To begin with, a (rather technical) definition is used to describe the leaves and double edges of $\mathcal{X}$.

Definition 3.3. The *parameters of* $\mathcal{X}$ are the integers described as follows:

1. Let $\mathcal{O} = \{y_0, \ldots, y_{n-1}\}$ be a γ-vertex-orbit of X which is a leaf in $\mathcal{X}$. If y_i is adjacent to y_{i+k} in X for some integer k, we say that k is *the parameter* of $\mathcal{O}$, or that k is *associated to* $\mathcal{O}$.
2. Suppose $\mathcal{O}_u = \{u_0, \ldots, u_{n-1}\}$ and $\mathcal{O}_v = \{v_0, \ldots, v_{n-1}\}$ are two γ-vertex-orbits of X such that there is a double edge between $\mathcal{O}_u$ and $\mathcal{O}_v$ in $\mathcal{X}$. Also, let us numerate u's and v's so that u_i and v_i are adjacent. Then u_i is also adjacent to v_{i-p} for some p called *the parameter* of the double edge $\mathcal{O}_u\mathcal{O}_v$.

Note that the parameter k of a leaf $\mathcal{O}$ does not depend on the numeration of the vertices of $\mathcal{O}$ and that $-k$ has essentially the same meaning as k. If p is the parameter of the double edge, say, $\mathcal{E} = \mathcal{O}_u\mathcal{O}_v$, then $-p$ is the parameter of $\mathcal{O}_v\mathcal{O}_u$: change of sign reverses the directed edge. However, any modification of the numeration within orbits $\mathcal{O}_u$, $\mathcal{O}_v$ may only swap p and $-p$.

The girth of the graph X plays an important role in this paper.

Proposition 3.4. 1. *If $\mathcal{X}$ contains two leaves $\mathcal{O}_x$, $\mathcal{O}_y$ with a common neighbour, then* $\text{girth}(X) \leq 12$. *If, additionally, k and l are the parameters of $\mathcal{O}_x$ and $\mathcal{O}_y$, respectively, and $l \equiv_n \pm k$, then* $\text{girth}(X) \leq 6$.
2. *If $\mathcal{X}$ contains a leaf $\mathcal{O}_x$ such that the edge $\mathcal{O}_x\mathcal{O}_u$ is adjacent to a double edge $\mathcal{O}_u\mathcal{O}_v$, then* $\text{girth}(X) \leq 10$. *If, additionally, k is the parameter of $\mathcal{O}_x$, p is the parameter of the double edge $\mathcal{O}_u\mathcal{O}_v$ and $p \equiv_n \pm k$, then girth$(X) \leq 5$.*

Proof. For the first part, see [11, p. 120]. For the second part, let $\mathcal{O}_x = \{x_i\}$, $\mathcal{O}_u = \{u_i\}$ and $\mathcal{O}_v = \{v_i\}$. It is readily established that

$$u_0x_0x_ku_kv_ku_{k+p}x_{k+p}x_pu_pv_0$$

is a 10-cycle in X. If also $k \equiv_n p$ (similarly, $-p$) holds, then X contains the 5-cycle $x_0x_{-k}u_{-k}v_{-k}u_0$. ■

A component of $\mathcal{X}^*$ isomorphic to I-tree, Y-tree, H-tree, A-tree, or B-tree will be called an *I-component*, *Y-component*, *H-component*, *A-component*, or *B-component*, respectively.

Proposition 3.5. *If C^* is a component of $\mathcal{X}^*$, then C^* is neither an A-component nor a B-component.*

Proof. If $\mathcal{X}^*$ contains an A- or a B-component, then X must be 5-arc-transitive. Namely, the diameter of such a component is 4 and hence, by Proposition 3.1, a 4-path in X is fixed vertex-wise by an involution. Thus X is 5-arc-transitive by Tutte's theorems. In particular, any 5-path can be carried onto any other 5-path by an automorphism.

Now suppose C^* is an arbitrary component of $\mathcal{X}^*$ which is an A-component (Fig. 5(a)). If $v = \{v_i\}$ is a leaf in $\mathcal{X}$ with parameter k, then every cycle containing the 5-path $y_0u_0v_0v_ku_kz_k$ must be longer than 12, thus contradicting Proposition 3.4, part 1. If v is not a leaf, then consider the 5-path $P = x_0y_0u_0v_0w_0t_0$. If t is a leaf in $\mathcal{X}$, then wt is an I-component of $\mathcal{X}^*$, and

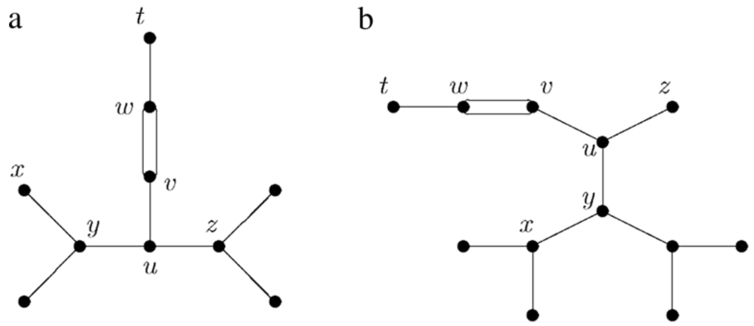

Fig. 5. *A*-trees, *B*-trees and 5-paths.

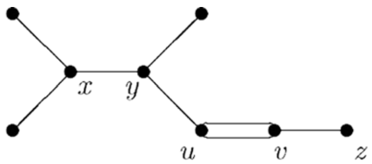

Fig. 6. A *H*-component and a 4-path.

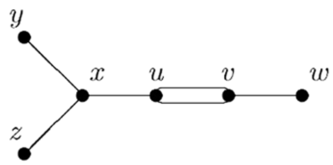

Fig. 7. A *Y*-component and a 3-path.

by Proposition 3.4, part 2, girth$(X) \leq 10$. But P is clearly not contained in a cycle of length ≤ 2. If t is not a leaf, then it is obvious that a cycle containing P must be of length at least 13 which contradicts Proposition 3.4, part 1.

For C^* being a B-component, the argument is similar to that of the previous paragraph. If C^* has a leaf v (in $\mathcal{X}$) with parameter k, we take the 5-path $y_0u_0v_0v_ku_kz_k$ (see Fig. 5(b)), and if C^* has no leaves, then we proceed with the 5-path $P = x_0y_0u_0v_0w_0t_0$. ■

Proposition 3.6. *$\mathcal{X}^*$ cannot have an H-component.*

Proof. Let C^* be an H-component of $\mathcal{X}^*$. By Proposition 3.1 and Tutte's theorems, X is at least 4-arc-transitive. Consider the 4-path $P = x_0y_0u_0v_0z_0$ in Fig. 6 where x, y, u belong to C^* (therefore, uv is a double edge). If z is a leaf in $\mathcal{X}$, then it follows from Proposition 3.4, part 2, that girth$(X) \leq 10$. However, P does not belong to any cycle of length ≤ 10. On the other hand, if z is not a leaf, then every cycle containing P must be longer than 12, contradicting Proposition 3.4, part 1. ■

4. Excluding *Y*-trees

It follows from Propositions 3.5 and 3.6 that $\mathcal{X}^*$ only has I- and/or Y-components. However, getting rid of Y's requires just a little more work. In this section we assume that $\mathcal{X}^*$ contains at least one Y-component. Proposition 3.1 then tells us that X is at least 3-transitive, hence we will explore 3-paths of X more in detail.

The following observation deals with some properties of the parameters of $\mathcal{X}$ which we will need in subsequent proofs.

Observation 4.1. *Let $0 \leq k, l, p < n$ such that $k, l, p \neq \frac{n}{2}$. Then it is easy to see that only one of the following conditions can hold:*

$$\begin{aligned} k - p &\equiv_n l \\ k - p &\equiv_n -l \\ -k - p &\equiv_n l \\ -k - p &\equiv_n -l. \end{aligned} \tag{1}$$

For example, if $k - p \equiv_n l$ and also $-k - p \equiv_n -l$, then by summing up both equalities we get $-2p \equiv_n 0$ and $p = \frac{n}{2}$, a contradiction.

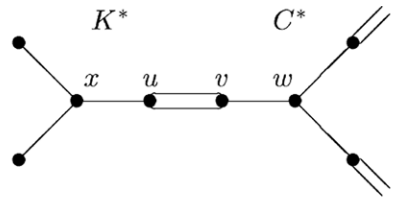

Fig. 8. A Y_2-component and a Y-component without a leaf.

Proposition 4.2. *Every leaf of $\mathcal{X}$ belongs to a Y-component.*

Proof. Suppose on the contrary, that a leaf is an endvertex of an I-component; then girth$(X) \leq 10$ by Proposition 3.4, part 2. Take a component C^* isomorphic to a Y-tree and choose the 3-path $P = x_0u_0v_0w_0$ (Fig. 7). If w is not a leaf of $\mathcal{X}$, or neither y nor z are leaves, then it is not difficult to see that any cycle containing P has length greater than 10. Hence, assume that w and (say) y are both leaves, and let k, l, p be the parameters of y, w, uv, respectively. The smallest cycle containing P must have length 10, so $l \not\equiv_n \pm p$ by Proposition 3.4, part 2. Also, it is clear that $k, l, p \neq \frac{n}{2}$. (If $p = \frac{n}{2}$, there would exist a 4-cycle $u_0v_0u_pv_p$.) Thus, the 3-path $R = v_0w_0w_lv_l$ is contained in two 10-cycles:

$$v_0w_0w_lv_lu_lv_{l-p}w_{l-p}w_{-p}v_{-p}u_0,$$
$$v_0w_0w_lv_lu_{l+p}v_{l+p}w_{l+p}w_pv_pu_p.$$

For P, in order to be contained in a 10-cycle, at least one of the six conditions

$$\begin{aligned} &k \equiv_n \pm l \\ &\pm k - p \equiv_n \pm l \end{aligned} \tag{2}$$

must be true. Suppose first that $k \not\equiv_n \pm l$. The observation (1) tells us that only one of the remaining four conditions can be true, so P is on at most one 10-cycle, while R is on two of them. Thus, necessarily $k \equiv_n l$ (so $k \not\equiv_n -l$, or vice versa) and there is a 10-cycle $x_0y_0y_kx_ku_kv_kw_kw_0v_0u_0$ which contains R, too. Hence R is on three 10-cycles and so is P. This means that one of the four conditions (2) must be true, say $-k - p \equiv_n l$, or, equivalently, $p \equiv_n -2k$. However, the 10-cycle

$$x_0y_0y_{-k}x_{-k}u_{-k}v_kw_kw_0v_0u_0$$

also contains R, hence R is on four cycles. As P cannot be on more than three due to (1), a contradiction follows. ■

The argument on the number of cycles containing a given path will be utilised often in this section, yet it will not be given to every detail as above.

Proposition 4.3. *There exists a Y-component of $\mathcal{X}^*$ which has two leaves in $\mathcal{X}$.*

Proof. Take an arbitrary path P in $\mathcal{X}$ having the maximal length, and denote one of its endvertices by u. Evidently, u is a leaf of $\mathcal{X}$ and so it lies on a Y-component C^*. Let x be the neighbour of u in C^*, and v, w the other two vertices of C^*. Then $x \in V(P)$, while v, w cannot both belong to P; say, $v \in V(P)$ and $w \notin V(P)$. If w were not a leaf, but connected to a vertex z by a double edge, then there would exist a path P' obtained from P by removing x, u and adding x, w, z whose length would be greater than that of P. Thus, C^* has two leaves, u and w. ■

A Y-component with two leaves in $\mathcal{X}$ will be called a *Y_2-component*. Hence, if $\mathcal{X}$ has Y-components, then it has Y_2-components. But what does the rest of $\mathcal{X}$ look like? We will find this out in a series of Propositions.

Proposition 4.4. *A Y_2-component and a Y-component without a leaf in $\mathcal{X}$ cannot be connected by a double edge in $\mathcal{X}$.*

Proof. Suppose K^* is a Y_2-component connected to a Y-component C^* via double edge in $\mathcal{X}$, and that C^* does not have a leaf (see Fig. 8). It is obvious that any cycle containing the 3-path $x_0u_0v_0w_0$ is longer than 12, but this is not possible due to Proposition 3.4, part 1. ■

Proposition 4.5. *A Y_2-component and a Y-component with exactly one leaf in $\mathcal{X}$ cannot be connected by a double edge in $\mathcal{X}$.*

Proof. Again, let K^* be the Y_2-component which is connected to a Y-component C^* by a double edge where C^* has just one leaf in $\mathcal{X}$ (Fig. 9(a)). Denote by k, l, m, p the parameters of the vertex-orbits y, z, s and the double edge uv, respectively. Choose two 3-paths in X : $P = x_0u_0v_0w_0$ and $R = x_0y_0y_kx_k$. The shortest cycle containing P must have length ≥ 12, so girth$(X) = 12$ by Proposition 3.4, part 1. It follows that $k \not\equiv_n \pm l$ and hence R is on two 12-cycles as described in [11, p. 120]:

$$x_0y_0y_kx_kz_kz_{k+l}x_{k+l}y_{k+l}y_lx_lz_lz_0$$
$$x_0y_0y_kx_kz_kz_{k-l}x_{k-l}y_{k-l}y_{-l}x_{-l}z_{-l}z_0.$$

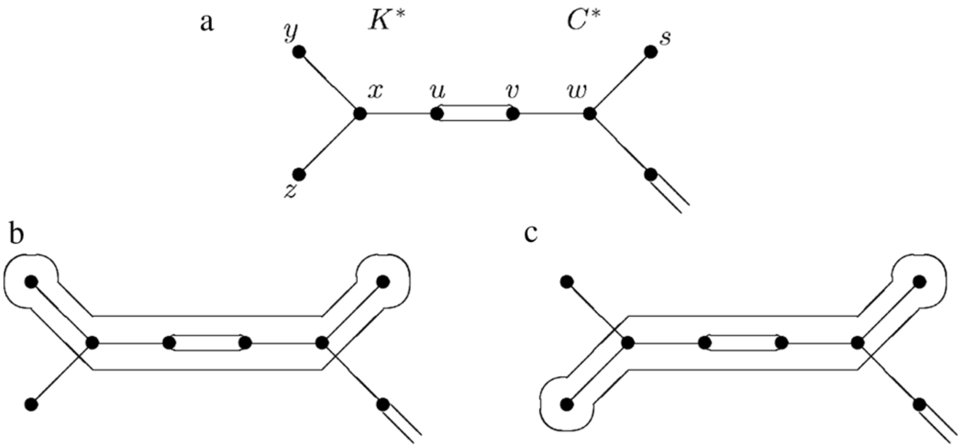

Fig. 9. Y-components, upper and lower cycles.

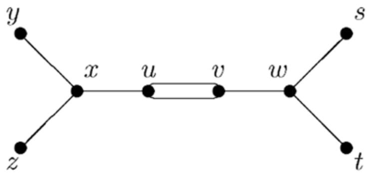

Fig. 10. Two Y_2-components.

These two 12-cycles will be called the *HB cycles*. We infer that every 3-path in X is on at least two 12-cycles. There are essentially two different possibilities for a 12-cycle to contain P (Fig. 9(b), (c)) which will be named an *upper* and a *lower* cycle. An upper cycle contains R (or, equivalently, $R^- = x_0y_0y_{-k}x_{-k}$). Similarly, a lower cycle contains the 3-path $R_z = x_0z_0z_lx_l$ (or $R_z^- = x_0z_0z_{-l}x_{-l}$). Thus, one of R, R^-, R_z, R_z^- is contained in three 12-cycles which means that every 3-path is on at least three 12-cycles. If there is an upper cycle containing P, then at least one of the following six conditions must hold true:

$$\begin{aligned} &k \equiv_n \pm m \\ &\pm k - p \equiv_n \pm m. \end{aligned} \tag{3}$$

For instance, if $k - p \equiv_n m$, then we have the upper cycle $x_0y_0y_kx_ku_kv_mw_ms_ms_0w_0v_0u_0$. Due to (1), at most one of the four conditions (3) can hold true. If $k \equiv_n m$ ($k \equiv_n -m$ leads to basically the same situation), then there are two upper cycles:

$$x_0y_0y_{\pm k}x_{\pm k}u_{\pm k}v_{\pm k}w_{\pm k}s_{\pm k}s_0w_0v_0u_0$$

(by the way, $k \not\equiv_n -m$ in this case). Therefore, P belongs to at most three upper cycles. Analogously, P belongs to at most three lower cycles. Because P is on at least three 12-cycles, there are four possibilities to check:

(i) P is on three upper 12-cycles. This means $k \equiv_n \pm m$ and $p \equiv_n \pm 2k$, say $k \equiv_n m$ and $p \equiv_n 2k$. Then there is an 8-cycle $u_kv_kw_ks_ks_0s_{\pm k}w_{\pm k}v_{\pm k}$ which is not possible.

(ii) P is on three lower 12-cycles. Similar argument as in the previous paragraph may be applied with the same conclusion.

(iii) P is on two upper cycles and one lower cycle. Then necessarily $k \equiv_n \pm m$, say $k \equiv_n m$. Since $l \not\equiv_n \pm m$ due to the girth of X, we have $p \equiv_n \pm l \pm k$, say $p \equiv_n k - l$. Now, the cycle $x_0y_0y_kx_ku_kv_kw_ks_ks_0w_0v_0u_0$ also contains R which belongs to two HB cycles as well. But R is contained in the following 12-cycle, too: $x_0y_0y_kx_ku_kv_lw_ls_ls_{l-k}w_{l-k}v_{l-k}u_0$. So each of R, P would be contained in four 12-cycles which means that P is on at least three upper cycles, but this leads to a contradiction as seen above.

(iv) P is on two lower cycles and one upper cycle. Following the steps as in the previous case ends in the same conclusion.

Thus, P does not belong to a 12-cycle which is impossible. ■

Proposition 4.6. *Two Y_2-components of $\mathcal{X}^*$ cannot be connected by a double edge in $\mathcal{X}$.*

Proof. If $\mathcal{X}^*$ has two Y_2-components that are connected by a double edge in $\mathcal{X}$, then $\mathcal{X}^*$ has no other components and so $\mathcal{X}$ is the graph depicted on Fig. 10. Similarly as in previous propositions, let us start with the 3-path $P = x_0u_0v_0w_0$. It is evident that every cycle through P has length at least 12, hence girth$(X) = 12$ by Proposition 3.4, part 1. Denote by k, l, m, o, p the parameters of the leaves y, z, s, t and of the double edge uv, respectively. Then the following restrictions

$$k \not\equiv_n \pm l$$
$$m \not\equiv_n \pm o$$
$$k, l, m, o \not\equiv_n \frac{n}{2}, \frac{n}{4}, \frac{n}{8}$$
$$p \not\equiv_n \frac{n}{2}, \frac{n}{4}$$

apply to these parameters. In order to have P on a 12-cycle, the graph X must satisfy at least one of the conditions collected in the following table:

	A	B	C	D
I.	$k \equiv_n \pm m$	$l \equiv_n \pm m$	$k \equiv_n \pm o$	$l \equiv_n \pm o$
II.	$\pm k \equiv_n \pm m + p$	$\pm l \equiv_n \pm m + p$	$\pm k \equiv_n \pm o + p$	$\pm l \equiv_n \pm o + p$

Now, suppose that one of the first-row conditions holds, for example, (I.A): $k \equiv_n m$. Then P is on two 12-cycles. However, one of them, $x_0y_0y_kx_ku_kv_kw_ks_ks_0w_0v_0u_0$, contains the 3-path $R = x_0y_0y_kx_k$ as well. Since there are two HB cycles containing R, we see that R is contained in three 12-cycles, and so is P. This means that additional condition from the table above must be true. But due to $k \equiv_n m$ neither (I.B) nor (I.C) can hold. Assume that (I.D) is true: $l \equiv_n o$. Then none of the second-row conditions holds, which can be proved as follows.

If (II.A) is true, then $p \equiv_n \pm 2k$ giving rise to the 8-cycle $x_{-k}y_{-k}y_0y_kx_ku_kv_{-k}u_{-k}$, a contradiction. In the same manner we dispose of (II.D). If, however, (II.B) is true, then $l \equiv_n k + p$ and there would be the 10-cycle $x_ky_ky_0x_0z_0z_lx_lu_lv_ku_k$ which is impossible. Accordingly, we can dismiss all cases (II.B) and (II.C).

Thus, we are left with $k \equiv_n m$ and $l \equiv_n o$, and no other conditions. It follows that P is contained in exactly four 12-cycles. Denote these four cycles by

$$Z = x_0y_0y_kx_ku_kv_kw_ks_ks_0w_0v_0u_0$$
$$W = x_0y_0y_{-k}x_{-k}u_{-k}v_{-k}w_{-k}s_{-k}s_0w_0v_0u_0$$
$$Z' = x_0z_0z_lx_lu_lv_lw_lt_lt_0w_0v_0u_0$$
$$W' = x_0z_0z_{-l}x_{-l}u_{-l}v_{-l}w_{-l}t_{-l}t_0w_0v_0u_0.$$

Also, let $Q = y_0x_0u_0v_0$ be another 3-path. Obviously, Q belongs to both Z and W, hence Q is contained in the 5-path $\bar{Q} = Z \cap W$. By 3-transitivity, there is an $\omega \in G$ such that $\omega(Q) = P$. Let $\bar{P} = \omega(\bar{Q})$; since $\bar{P}$ contains P, it is a 5-path contained in an intersection of two of the four cycles Z, W, Z', W'. Clearly $\bar{P} = Z \cap W$ or $\bar{P} = Z' \cap W'$. However, $\bar{Q} - Q$ is a 2-path while $\bar{P} - P$ is not. Therewith we have shown that two of the first-row conditions cannot be true at the same time.

Thus, $k \equiv_n m$ and no other first-row conditions hold. For R, it is not difficult to see that there are four 12-cycles: Z, the two HB cycles, and $x_0y_0y_kx_ku_kv_{k-p}w_{k-p}s_{k-p}s_{-p}w_{-p}v_{-p}u_0$, all of them containing R. It follows that P belongs to four 12-cycles, too, hence some second-row conditions must be true. It is easy to see that only (II.D) is possible, so $l \equiv_n o + p$ and $-l \equiv_n -o + p$, or $-l \equiv_n o + p$ and $l \equiv_n -o + p$. But then $2p \equiv_n 0$, contrary to the restrictions above. This means that no first-row conditions can hold.

Let us deal with the possibility that only second-row conditions may be true. However, in each of the sets (II.A), (II.B), (II.C), (II.D), only one condition can hold. This can be proved analogously as at the end of the previous paragraph: if $k \equiv_n -m + p$ and $-k \equiv_n m + p$, then $2p \equiv_n 0$, etc. Since R is contained in two HB cycles, P is on (at least) two 12-cycles, hence two second-row conditions must hold. If, say, $k \equiv_n m + p$, then there is the 12-cycle $x_0y_0y_kx_ku_kv_mw_ms_ms_0w_0v_0u_0$ which also contains R; therefore, R (and P, too) belongs to three 12-cycles. It follows that except for (II.A), two other second-row conditions hold. If (II.C) holds, then R is on four 12-cycles; if (II.C) does NOT hold, then both (II.B) and (II.D) hold, and the 3-path $R_z = x_0z_0z_lx_l$ is on four 12-cycles. This means that P (as any other 3-path) must belong to four 12-cycles, so (II.A), (II.B), (II.C), and (II.D) must all be true.

Since $k \equiv_n m + p$, then either $l \equiv_n -m + p$ or $l \equiv_n m - p$ (assume the former). Accordingly, either $k \equiv_n o - p$ or $k \equiv_n -o + p$ (assume the latter here). It follows that $l \equiv_n -o - p$ necessarily. Yet, from these four conditions we infer that $4p \equiv_n 0$, contrary to the restrictions above. In other subcases we conclude in the same way. Hence the 3-path P does not belong to any 12-cycle, which is absurd. ■

Even the last resort for a Y_2-component to be contained in $\mathcal{X}^*$ fails, because such a component "neighbouring" an I-component does not occur, as we shall see below.

Proposition 4.7. *A Y_2-component and an I-component cannot be connected by a double edge in $\mathcal{X}$.*

Proof. Put $P = x_0u_0v_0w_0$ (Fig. 11) and denote by k, l, p, q the parameters of the vertices y, z and of the edges uv, ws, respectively. It is clear that the shortest cycle containing P must be of length at least 11. Thus, by Proposition 3.4, part 1,

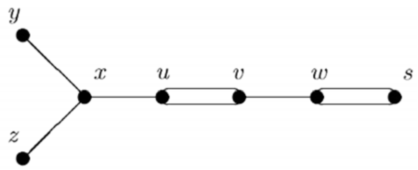

Fig. 11. A Y-component and an I-component.

the graph surely contains 12-cycles (namely, the HB cycles), so P belongs to a 12-cycle. Also, the following set of restrictions apply to these parameters:

$$k \not\equiv_n \pm l$$
$$k, l, p, q \not\equiv_n \frac{n}{2}.$$

As in Proposition 4.5, the two possibilities for such a 12-cycle will be called an *upper* (traversing vertices in y) and a *lower* (traversing those in z) cycle. Therefore, if there is an upper cycle, then one of the following conditions must hold true:

$$2k \equiv_n \pm q$$
$$\pm 2k \equiv_n \pm q + p.$$

(A) $2k \equiv_n q$ (or $-q$, which is only formally different). Then P belongs to two upper 12-cycles: $Z = x_0y_0y_ky_{2k}x_{2k}u_{2k}v_{2k}w_{2k}s_{2k}w_0v_0u_0$ and $W = x_0y_0y_{-k}y_{-2k}x_{-2k}u_{-2k}v_{-2k}w_{-2k}s_0w_0v_0u_0$. It also follows that $2k \not\equiv_n -q$, $2k \not\equiv_n q+p$ and $-2k \not\equiv_n -q+p$. If $-2k \equiv_n q+p$, then there would exist additional 12-cycle $Z^* = x_0y_0y_ky_{2k}x_{2k}u_{2k}v_{2k}w_{2k}s_{2k}w_{-p}v_{-p}u_0$. But the symmetric difference of Z and Z^* would be a 10-cycle, which is not possible because girth$(X) \geq 11$. Same conclusion is obtained if $2k \equiv_n -q+p$. So Z, W are the only two upper 12-cycles, and they have exactly a 4-path in common.

(B) $2k \not\equiv_n \pm q$, $2k \equiv_n q+p$. Then P belongs to the upper cycle $W' = x_0y_0y_ky_{-2k}x_{-2k}u_{-2k}v_{-2k}w_{-2k}s_{-2k}w_{-p}v_{-p}u_0$, and also $2k \not\equiv_n -q+p$ and $-2k \not\equiv_n \pm q+p$ holds. In this case there is only one upper cycle.

The same conclusion can be made for lower cycles, with l instead of k. However, because $k \not\equiv_n \pm l$, (A) cannot hold for both upper and lower cycles.

Now suppose (A) holds for, say, upper cycles. The two upper cycles have exactly a 4-path in common. But the 3-path $R = x_0y_0y_kx_k$ belongs to two HB cycles which have exactly a 5-path in common. Therefore, P is also contained in a lower 12-cycle and hence (B) holds for lower cycles, for example, $2l \equiv_n q + p$ which gives rise to $W'' = x_0z_0z_{-l}z_{-2l}x_{-2l}u_{-2l}v_{-2l}w_{-2l}s_{-2l}w_{-p}v_{-p}u_0$. But $W'' \cap Z$ is a 6-path and $W'' \cap W$ is a 3-path which is not possible. (The conclusion is the same for other (B) conditions.) Therefore, (A) is false for upper cycles; nevertheless, the same procedure tells us that (A) is also false for lower cycles.

It remains to see what happens if (B) holds. Clearly, (B) must be true for both upper and lower cycles. In the same manner as before we infer that these two 12-cycles have either a 6-path or a 3-path in common, a contradiction. ■

The last four propositions combined together give us the following corollary.

Corollary 4.8. *The graph $\mathcal{X}^*$ cannot have Y-components.*

Therefore, Propositions 3.2, 3.5, 3.6 and Corollary 4.8 combined together give us Theorem 1.3, whereas Theorem 1.4 follows directly from [6,11] and Theorem 1.3.

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